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Massive multi-flavor Schwinger model at finite temperature and on compact space[1]

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The multi-flavor Schwinger model on R^1 at finite temperature T is mathematically equivalent to the model on S^1 at $T=0$. The latter is reduced to a quantum mechanical system of $N-1$ degrees of freedom. Physics sensitively depends on the parameter m/T. Finite temperature behavior of the massive Schwinger model is quite different from that of the massless Schwinger model.

1. Introduction

The Schwinger model is QED in two dimensions, described by

$$\mathcal{L} = -\tfrac{1}{4}\, F_{\mu\nu}F^{\mu\nu} + \sum_{a=1}^{N} \overline{\psi}_a \Big\{ \gamma^\mu (i\partial_\mu - eA_\mu) - m_a \Big\} \psi_a \quad . \tag{1}$$

Massless theory ($m_a = 0$) is exactly solvable. In the $N=1$ (one flavor) model the gauge boson acquires a mass without breaking the gauge invariance.[1] It has the θ vacuum and a non-vanishing chiral condensate $\langle\, \overline{\psi}\psi \,\rangle \neq 0$.[2]

The $N \geq 2$ model is distinctively different from the $N=1$ model. The spectrum contains $N-1$ massless bosons. The chiral condensate vanishes, $\langle\, \overline{\psi}\psi \,\rangle = 0$,[3] as in two dimensions continuous symmetry, $SU(N)$ chiral symmetry in this case, cannot be spontaneously broken. Nonvanishing $\langle\, \overline{\psi}\psi \,\rangle$ in the N=1 theory is allowed, since the $U(1)$ chiral symmetry is broken by an anomaly.

With massive fermions the model is not exactly solvable. The effect of the fermion mass in the $N=1$ theory is minor for $m/e \ll 1$, except that it necessiates the θ vacuum. For $N \geq 2$ the situation is quite different. Coleman showed[4] that in the $N=2$ model with $m_a = m$ two resultant bosons acquire masses given by

$$\mu_1 \sim \frac{\sqrt{2}\, e}{\sqrt{\pi}} \quad , \quad \mu_2 \propto m^{2/3} e^{1/3} \Big| \cos \frac{1}{2}\theta \Big|^{2/3} \quad . \tag{2}$$

Surprising is the fractional power dependence on m, e and $|\cos \frac{1}{2}\theta|$ of the second boson mass μ_2, resulting from the self-consistent re-alignment of the vacuum against fermion masses. The effect of fermion masses is always non-perturbative even if $m/e \ll 1$.

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How does μ_2 depend on m in the N flavor model? How does it change at finite temperature? Does the chiral condensate vanish at suffieciently high temperature? Can the fermion mass be treated as a small perturbation at high temperature?

These are questions addressed in this article. We present a powerful method to evaluate various physical quantities at zero and finite temperature. We shall recognize the importance of a dimensionless parameter m/T. The behavior at $m/T \gg 1$ is quite different from that at $m/T \ll 1$. Coleman's result (2) corresponds to $m/T \gg 1$ as $T \to 0$. At high temperature $m/T \ll 1$, one finds $\mu_2 \propto m$, a result obtained in peturbation theory. This work is based on the result obtained in ref. 5.

2. At finite temperature and on a circle

In Matsubara's formalism the model at finite temperature T in equilibrium is equivalent to an Euclidean field theory, or a theory with an imaginary time τ, satisfying boundary conditions

$$\psi_a(\tau + \frac{1}{T}, x) = -\psi_a(\tau, x) \quad , \quad A_\mu(\tau + \frac{1}{T}, x) = A_\mu(\tau, x) \quad . \tag{3}$$

If one, instead, places the model on a circle of circumference L (at zero temperature) with boundary conditions

$$\psi_a(t, x + L) = -\psi_a(t, x) \quad , \quad A_\mu(t, x + L) = A_\mu(t, x) \quad , \tag{4}$$

then one obtains a theory which is, after Wick's rotation, mathematically equivalent to the finite temperature field theory defined by (3). Various physical quantities in the Schwinger model at $T \neq 0$ are related to corresponding ones in the model on S^1 by substitution of T by L^{-1}.

Our strategy is to solve the model on a circle S^1 with an arbitrary size L. There is powerful machinery which specifically works on S^1.[5–8]

3. Reduction to a quantum mechanical system

Fermion operators on S^1 can be expressed in terms of bosonic operators: Take $\gamma^\mu = (\sigma_1, i\sigma_2)$ and write $\psi_a^T = (\psi_+^a, \psi_-^a)$. In the interaction picture defined by free massless fermions

$$\psi_\pm^a(t, x) = \frac{1}{\sqrt{L}}\, C_\pm^a\, e^{\pm i\{q_\pm^a + 2\pi p_\pm^a (t \pm x)/L\}} : e^{\pm i\sqrt{4\pi}\phi_\pm^a(t,x)} : \tag{5}$$

$$e^{2\pi i p_\pm^a} \,|\,\text{phys}\,\rangle = \,|\,\text{phys}\,\rangle \quad . \tag{6}$$

The Klein factors are given by $C_+^a = e^{i\pi\sum_{b=1}^{a-1}(p_+^b + p_-^b)}$ and $C_-^a = e^{i\pi\sum_{b=1}^{a}(p_+^b - p_-^b)}$. Here $[q_\pm^a, p_\pm^b] = i\,\delta^{ab}$ and $\phi_\pm^a(t,x) = \sum_{n=1}^\infty (4\pi n)^{-1/2}\,\{c_{\pm,n}^a\, e^{-2\pi i n(t\pm x)/L} + \text{h.c.}\}$ where $[c_{\pm,n}^a, c_{\pm,m}^{b,\dagger}] = \delta^{ab}\delta_{nm}$. The : : indicates normal ordering with respect to $(c_n, c_n^\dagger)$. The antiperiodic boundary condition is ensured by a physical state condition (6). In physical states $p_\pm^a$ takes integer eigenvalues.

After substituting the bosonization formula (5), the total Hamiltonian in the Schrödinger picture becomes

$$
\begin{aligned}
H_{\rm tot} &= H_0 + H_\phi + H_{\rm mass} \\
H_0 \ &= -\frac{e^2 L}{2}\frac{d^2}{d\Theta_W^2} + \frac{\pi}{2L}\sum_{a=1}^N \Big\{ (p_+^a - p_-^a)^2 + (p_+^a + p_-^a + \frac{\Theta_W}{\pi})^2 \Big\} \\
H_\phi &= \int_0^L dx\, \frac{1}{2}\Big[\sum_{a=1}^N \Big\{ \Pi_a^2 + (\phi_a')^2 \Big\} + \frac{e^2}{\pi}\Big(\sum_{a=1}^N \phi_a \Big)^2 \Big]
\end{aligned} \tag{7}
$$

Here $p_+^a - p_-^a$ and $p_+^a + p_-^a$ correspond to the charge and chiral charge operators, respectively. Θ_W is the phase of the Wilson line around the circle, the only physical degreee of freedom of the gauge field on the circle, $A_1 = \Theta_W(t)/eL$. The coupling between $p_\pm^a$ and Θ_W is induced through the chiral anomaly.[7] $\phi_a = \phi_+^a + \phi_-^a$ and Π_a is its canonical conjugate. $H_{\rm mass}$ is the fermion mass term.

Notice that (7) is an exact operator identity. In the absence of fermion masses $H_{\rm tot} = H_0 + H_\phi$. The zero modes $(\Theta_W, q_\pm^a)$ decouple from the oscilatory modes ϕ_a, and the Hamiltonian is exactly solvable. The spectrum contains one massive field $N^{-1/2}\sum_{a=1}^N \phi_a$ with a mass $\mu = (N/\pi)^{1/2} e$, and $N-1$ massless fields.

To examine effects of $H_{\rm mass}$, first note that $H_{\rm mass}$, and therefore $H_{\rm tot}$, commutes with $p_+^a - p_-^a$. Hence we can restrict ourselves to states with $(p_+^a - p_-^a)\,|\,{\rm phys}\,\rangle = 0$. With this restriction a complete set of eigenfunctions and eigenvalues of H_0 is

$$
\begin{aligned}
\Phi_s^{(n_1,\cdots,n_N)} \ &= \ \frac{1}{(2\pi)^N}\, u_s[\Theta_W + \frac{2\pi}{N}\sum_a n_a]\, e^{i\sum n_a(q_+^a + q_-^a)} \\
E_s^{(n_1,\cdots,n_N)} \ &= \ \mu(s+\tfrac{1}{2}) + \frac{2\pi}{L}\sum_{a=1}^N n_a^2 - \frac{2\pi}{NL}\Big\{ \sum_{a=1}^N n_a \Big\}^2
\end{aligned} \tag{8}
$$

where a harmonic oscillator wave function u_s satisfies $\frac{1}{2}(-\partial_x^2 + x^2)u_s = (s+\frac{1}{2})u_s$ with $x = (\pi e^2 L^2/N)^{-1/4}\Theta_W$. The ground states of H_0 are infinitely degenerate for $n_1 = \cdots = n_N$ due to the invariance under a large gauge transformation $\Theta_W \to \Theta_W + 2\pi$ and $\psi_a \to e^{2\pi i x/L}\psi_a$.

$H_{\rm mass}$ induces transitions among $\Phi_s^{(n_1,\cdots,n_N)}$'s. It also gives finite masses to the $N-1$ previously massless fields. The structure of the vacuum sensitively depends on $H_{\rm mass}$. The effect of $H_{\rm mass}$ turns out quite nonperturbative so long as $m_a \neq 0$.

It is more convenient to work in a coherent state basis given by

$$
\Phi_s(\varphi_a;\theta) = \frac{1}{(2\pi)^{N/2}} \sum_{\{n,r_a\}} e^{in\theta + i\sum_{a=1}^{N-1} r_a\varphi_a}\ \Phi_s^{(n+r_1,\cdots,n+r_{N-1},n)} \tag{9}
$$

Transitions in the s index may be ignored to a very good approximation. We seek the vacuum in the form

$$
\Phi_{\rm vac}(\theta) = \int_0^{2\pi} d\varphi_1\cdots d\varphi_{N-1}\ f(\varphi_a;\theta)\ \Phi_0(\varphi_a;\theta)\ . \tag{10}
$$

Let $\chi_\alpha = U_{\alpha a}\phi_a$ and μ_α be a mass eigenstate field and its mass. Then matrix elements of $H_{\rm mass}$ in the coherent state basis are

$$\begin{aligned}
&\langle\, \Phi_0(\varphi'_a;\theta') | H_{\rm mass} | \Phi_0(\varphi_a\theta)\,\rangle = -\delta_{2\pi}(\theta'-\theta) \prod_{b=1}^{N-1} \delta_{2\pi}(\varphi'_b - \varphi_b) \sum_{a=1}^{N} A_a \cos\varphi_a \\
&A_a = 2m_a e^{-\pi/N\mu L} \prod_{\alpha=1}^{N} B(\mu_\alpha L)^{(U_{\alpha a})^2} \quad , \quad \varphi_N = \theta - \sum_{a=1}^{N-1} \varphi_a \quad . \qquad (11)
\end{aligned}$$

$B(z)$ is given by

$$\begin{aligned}
B(z) &= \exp\Big\{ \sum_{n=1}^{\infty} \Big(\frac{1}{n} - \frac{1}{\sqrt{n^2 + (z/2\pi)^2}} \Big) \Big\} \\
&= \frac{z}{4\pi} \exp\Big\{ \gamma + \frac{\pi}{z} - 2\int_1^\infty \frac{du}{(e^{uz}-1)\sqrt{u^2-1}} \Big\} \ . \qquad (12)
\end{aligned}$$

The eigenvalue equation $(H_0 + H_{\rm mass})\, \Phi_{\rm vac}(\theta) = E\, \Phi_{\rm vac}(\theta)$ becomes

$$\begin{aligned}
&\Big\{ - \triangle_N^\varphi + V_N(\varphi) \Big\}\ f(\varphi) = \epsilon\ f(\varphi) \\
&\triangle_N^\varphi = \sum_{a=1}^{N-1} \frac{\partial^2}{\partial \varphi_a^2} - \frac{2}{N-1} \sum_{a<b}^{N-1} \frac{\partial^2}{\partial\varphi_a \partial\varphi_b} \\
&V_N(\varphi) = - \frac{NL}{2(N-1)\pi}\ \sum_{a=1}^{N} A_a \cos\varphi_a \quad . \qquad (13)
\end{aligned}$$

Here $\epsilon = NEL/2\pi(N-1)$. Eq. (13) is nothing but the Schrödinger equation with the kinetic and potential terms given by $-\triangle_N^\varphi$ and $V_N(\varphi)$, respectively.

The potential $V_N(\varphi)$ depends, through A_α defined in (11), on μ_α and $U_{\alpha a}$ which are to be self-consistently determined from the ground state wave function $f(\varphi_a;\theta)$ of the Schrödinger equation (13). μ_α's and $U_{\alpha a}$'s are determined by

$$\begin{aligned}
&\frac{\mu^2}{N} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} + \begin{pmatrix} R_1 & & \\ & \ddots & \\ & & R_N \end{pmatrix} = U^T \begin{pmatrix} \mu_1^2 & & \\ & \ddots & \\ & & \mu_N^2 \end{pmatrix} U \\
&R_a = \frac{4\pi}{L}\, A_a \langle\, \cos\varphi_a\, \rangle_f = -4\pi m_a \,\langle\, \overline{\psi}_a \psi_a\, \rangle_\theta \quad . \qquad (14)
\end{aligned}$$

We have denoted $\langle\, g(\varphi)\, \rangle_f = \int [d\varphi]\, g(\varphi) |f(\varphi)|^2$. We need to solve (11), (13), and (14) self-consistently.

We have shown that the N flavor massive Schwinger model is reduced to quantum mechanics of $N-1$ degrees of freedom in which the potential has to be fixed self-consistently with its ground state wave function.

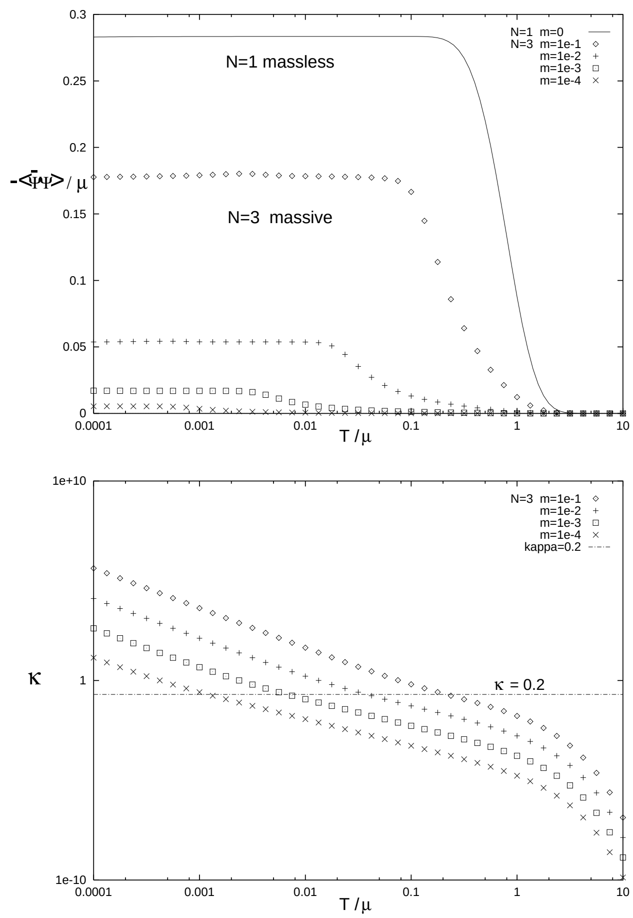

Figure 1: Chiral condensate $-\langle\, \overline{\psi}\, \psi\, \rangle/\mu$ and κ_0 as a function of temperature T at $\theta = 0$ in the $N = 1$ and $N = 3$ models. In the $N = 3$ model $m/\mu = 10^{-1}, 10^{-2}, 10^{-3}$ and 10^{-4}. The crossover in $\langle\, \overline{\psi}\, \psi\, \rangle/\mu$ takes place when $\kappa_0 \sim 0.2$.

4. N=1 (one flavor)

One flavor case is special. $\varphi_1 = \theta$ and there is only one massive boson with a mass μ_1 . This case was analysed in detail in refs. 7 and 9.

The vacuum on S^1 is $\Phi_{\rm vac}(\theta) = \Phi_0(\theta)$. Converting all expressions to finite temperature case, we find the boson mass and chiral condensate to be

$$\begin{aligned} \mu_1^2 &= \mu^2 + 8\pi m T\, \cos\theta\, e^{-\pi T/\mu}\, B\Big(\frac{\mu_1}{T}\Big) \\ \langle\, \overline{\psi}\, \psi\, \rangle_\theta &= -2T\, \cos\theta\, e^{-\pi T/\mu}\, B\Big(\frac{\mu_1}{T}\Big) \end{aligned} \tag{15}$$

where $\mu = e/\sqrt{\pi}$. At $T = 0$

$$\begin{aligned} \mu_1 &= \sqrt{\mu^2 + m^2 e^{2\gamma} \cos^2\theta} + m e^{\gamma} \cos\theta \\ \langle\, \overline{\psi}\, \psi\, \rangle_\theta &= -\frac{e^{\gamma}}{2\pi}\, \mu_1 \cos\theta\ . \end{aligned} \tag{16}$$

Notice that the condensate is nonvanishing even for $m = 0$. As T is raised, it shows a crossover around $T = \mu$, and approaches zero at high temperature. See fig. 1. The correction due to the fermion mass $m \ll \mu$ is minor.

5. N≥2 (multi-flavors) at low and high T

The situation in the multi-flavor case is quite different. When $m_a = 0$, $V_N(\varphi) = 0$ and $f(\varphi)$ =const so that $\langle \cos\varphi_a \rangle_f = 0$. Consequently $\langle \overline{\psi}_a \psi_a \rangle_\theta = 0$.

Suppose that fermion masses are degenerate: $m_a = m$. There results one heavy boson and $N-1$ light bosons with masses μ_1 and μ_2, respectively. The potential in (13) becomes

$$\begin{aligned} V_N(\varphi) &= \; - \kappa_0 \sum_{a=1}^{N} \cos\varphi_a \\ \kappa_0 &= \frac{N}{(N-1)\pi} \, \frac{m}{T} \, B\Big(\frac{\mu_1}{T}\Big)^{\frac{1}{N}} B\Big(\frac{\mu_2}{T}\Big)^{1-\frac{1}{N}} \, e^{-\pi T/N\mu} \end{aligned} \tag{17}$$

where μ_1 and μ_2 are determined by

$$\begin{aligned} \mu_1^2 &= \mu^2 + \mu_2^2 \\ \mu_2^2 &= \frac{8\pi^2(N-1)}{N} \, \kappa_0 T^2 \, \langle \cos\varphi \rangle_f = -4\pi m \langle \overline{\psi}_a \psi_a \rangle_\theta \quad . \end{aligned} \tag{18}$$

Recognize that two parameters κ_0 and θ fix the potential $V_N(\varphi)$.

If $m \neq 0$, κ_0 becomes very large at low temperature $T \to 0$. In this regime the potential term dominates over the kinetic energy term in the Schrödinger equation (13). The wave function $f(\varphi)$ has a sharp peak around the minimum of the potential. The minimum is located at $\varphi_a = \bar\theta/N$ where $\bar\theta = \theta - 2\pi[(\theta/2\pi) + \frac{1}{2}]$. As θ varies from $-\pi$ to $+\pi$, the minimum moves from $\varphi_a = -\pi/N$ to $\varphi_a = +\pi/N$, and jumps back to $\varphi_a = -\pi/N$.

In the $T \to 0$ limit, $\langle \cos\varphi \rangle_f = \cos\bar\theta$ so that

$$\frac{1}{\mu} \langle \overline{\psi} \psi \rangle_\theta = -\frac{1}{4\pi} \Big(2e^\gamma \cos\frac{\bar\theta}{N} \Big)^{\frac{2N}{N+1}} \Big(\frac{m}{\mu}\Big)^{\frac{N-1}{N+1}} \quad \text{for} \;\; T \ll m^{\frac{N}{N+1}} \mu^{\frac{1}{N+1}} \; . \tag{19}$$

Two important observations follow. First, the dependence of the condensate on m is non-analytic. It has fractional power dependence. The effect of fermion masses is nonperturbative in this limit. Secondly, as a function of θ, the condensate has a cusp at $\theta = \pm\pi$, which originates from the discontinuous jump in the location of the minimum of the potential.

In the opposite limit $\kappa_0 \ll 1$, which includes both $m \to 0$ (with $T > 0$ kept fixed) and $T \gg \mu$, the potential $V_N(\varphi)$ can be treated as a small peturbation in (13). One finds

$$\langle \cos\varphi_a \rangle_f = \begin{cases} (1+\cos\theta)\kappa_0 & \text{for } N = 2 \\ \kappa_0 & \text{for } N \geq 3. \end{cases} \tag{20}$$

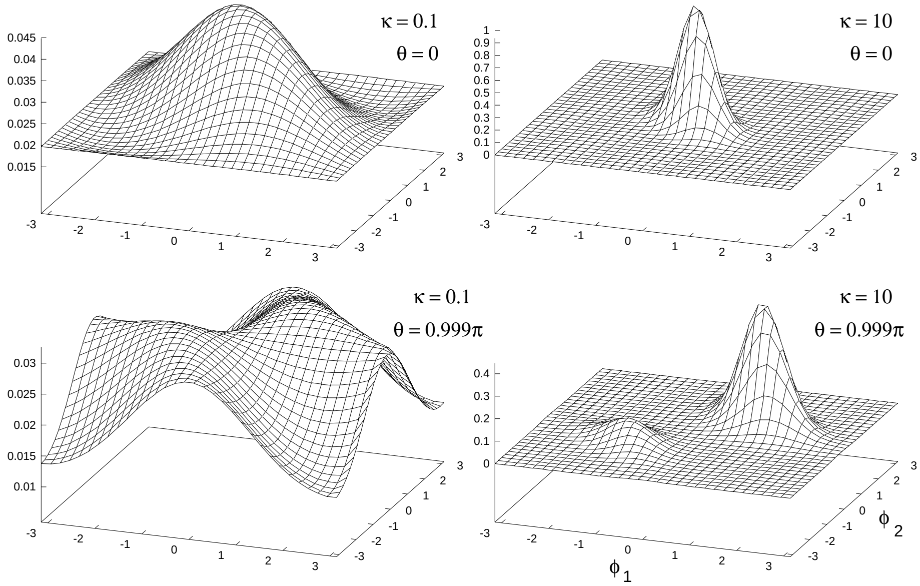

Figure 2: Wave function $|f(\varphi)|^2$ in the $N=3$ model.

There appears no θ-dependence for $N \geq 3$ to this order. The condensate for $N \geq 3$ is found to be

$$\frac{1}{\mu}\,\langle\,\overline{\psi}\,\psi\,\rangle_\theta = -\frac{2N}{\pi(N-1)}\,\frac{m}{\mu}\,\begin{cases}\left(\dfrac{\mu e^\gamma}{4\pi T}\right)^{2/N} & \text{for } m^{\frac{N}{N+1}}\mu^{\frac{1}{N+1}} \ll T \ll \mu \\ e^{-2\pi T/N\mu} & \text{for } T \gg \mu\end{cases} \tag{21}$$

For $N=2$, the expressions for $\langle\,\overline{\psi}\,\psi\,\rangle_\theta$ in (21) must be multiplied by a factor $2\cos^2\frac{1}{2}\theta$. Notice that the condensate is linearly proportional to m in this regime.

6. Crossover in $\langle\,\overline{\psi}\psi\,\rangle_{\theta,T}$

For general values of T and m, we need to solve the set of equations numerically. In the two flavor case the Schrödinger equation is

$$\left\{-\frac{\partial^2}{\partial\varphi_1^2} - 2\kappa_0\cos\tfrac{1}{2}\theta\cos(\varphi_1 - \tfrac{1}{2}\theta)\right\} f(\varphi_1) = \epsilon\, f(\varphi_1)\ . \tag{22}$$

This is the equation for a quantum pendulum. The strength of the potential is $2\kappa_0\cos\frac{1}{2}\theta$, which changes the sign at $\theta=\pm\pi$. In particular, the potential vanishes at $\theta=\pm\pi$ and $f(\varphi_1)$ =constant. Consequently the chiral condensate vanishes at θ=$\pm\pi$. This is a special feature of $N=2$.

For $N=3$ the equation is

$$\left\{-\frac{\partial^2}{\partial\varphi_1^2} - \frac{\partial^2}{\partial\varphi_2^2} + \frac{\partial^2}{\partial\varphi_1\partial\varphi_2} - \kappa_0\Big[\cos\varphi_1 + \cos\varphi_2 + \cos(\varphi_1+\varphi_2-\theta)\Big]\right\} f(\varphi) = \epsilon\, f(\varphi)\ . \tag{23}$$

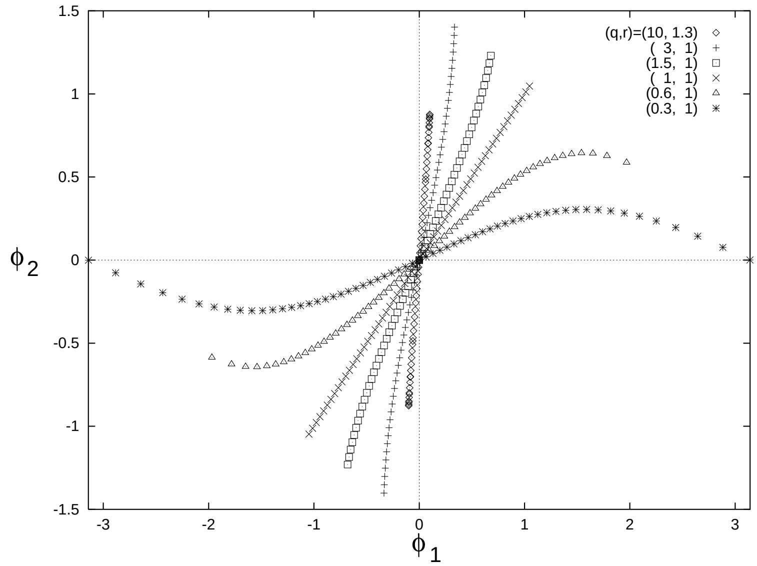

Figure 3: The location of the minimum of the potential in the $N = 3$ model is displayed with various values of (q, r) in (23). For (q, r)=(10, 1.3) the minimum at θ=$\pm\pi$ is located at the origin.

The potential term never vanishes unless $\kappa_0 = 0$, or equivalently $m = 0$ or $T \to \infty$. The equation can be solved numerically for an arbitrary κ_0. The ground state wave function $|f(\varphi)|^2$ has been displayed for various κ_0 and θ in fig. 2.

With given values of m and T the chiral condensate is determined by solving (17), (18), and (23) simultaneously. We developed an iteration procedure which yields a consistent set of values of m/μ, T/μ, μ_a/μ, and κ_0. It takes less than ten iterations even for moderate values of $\kappa_0 \sim 1$ to achieve four digits accuracy.

In the top figure of fig. 1 we have displayed the condensate $\langle\, \overline{\psi}\,\psi\,\rangle_{\theta=0}/\mu$ as a function of T/μ and m/μ in the N=3 case too. With a given m/μ the condensate is almost constant at low T/μ, and sharply drops to a small value around T_*. This crossover takes place when $\kappa_0(m, T_*) \sim 0.2$ for a wide range of the value of m/μ. In the bottom figure we have shown a plot for κ_0 in the same region of the m-T space.

The asymptotic formulas (19) and (21) are quite accurate for $\kappa_0 > 1$ and $\kappa_0 < 0.1$. The important parameter is κ_0.

7. The singularity at $\theta = \pm\pi$ and fermion masses

As shown in (19), the chiral condensate at $T = 0$ shows a cusp singularity in its θ dependence when fermion masses are degenerate. When fermion masses are not degenerate, the potential $V_N(\varphi)$ in (13) is deformed from the symmetric one in (17). Accordingly the location of the minimum is shifted. The cusp singularity appears when the location of the minimum of the potential makes a discontinuous jump.

With given $\{m_a\}$, A_a in $V_N(\varphi)$ is determined self-consistently. We have determined the location of the minimum of the potential in the three flavor case. The potential is

proportional to

$$F(\varphi_1, \varphi_2; \theta) = -q \cos\varphi_1 - r \cos\varphi_2 - \cos(\theta - \varphi_1 - \varphi_2) \tag{24}$$

where $q = A_1/A_3$ and $r = A_2/A_3$.

In the symmetric case q=r=1 the location of the minimum moves from $(-\frac{1}{3}\pi, -\frac{1}{3}\pi)$ to $(+\frac{1}{3}\pi, +\frac{1}{3}\pi)$ as θ varies from $-\pi$ to $+\pi$, and makes a jump. At θ=0 the minimum is located at the origin for arbitrary (q, r). We have plotted trajectories of the location of the minimum for several typical values of (q, r) in fig. 3.

So long as asymmetry is small, there is a discontinuous jump at $\theta = \pm\pi$. However, a sufficiently large asymmetry restores continuity. For instance, with $(q < 0.5, r = 1)$ the minimum at $\theta = \pm\pi$ is located at $(\varphi_1, \varphi_2) = (\pm\pi, 0)$. The trajectory makes a closed loop on the φ_1-φ_2 torus. For $(q \gg 1, r = 1)$, the minimum at $\theta = \pi$ is around $(0, \frac{1}{2}\pi)$ so that the discontinuity remains. However, if one adds a small aysmmetry in "r", the minimum is pushed back to the origin. For instance, for $(q, r) = (10, 1.3)$, the minimum starts to turn back at $\theta \sim 0.8\pi$ and reaches the origin at $\theta = \pi$. The implication to QCD physics is profound. As $m_s \gg m_d > m_u$, we are facing at a case $q \gg r > 1$ in which there is no singularity in θ any more.[10]

We conclude that a sufficiently large asymmetry in the fermion masses removes the cusp singularity at $\theta = \pm\pi$ in $\langle\, \overline{\psi}\psi \,\rangle$ at $T = 0$.

Acknowledgments

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